Math 113 Fall 09 Exam 1 Key

1. d
2. c
3. a
4. b
5. f
6. d
7. c
8. e
9. (a) 2
   (b) 78
   (c) 72 ft-lbs
   (d) $x \ln x - x + C$
   (e) $x \ln^2 x - 2x \ln x + 2x + C$
   (f) $-(1+x)e^{-x} + C$
   (g) $\frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$
   (h) $\frac{\pi}{2}$
10. We can use the half angle identity:

$$\int_0^\pi \sin^4 x\,dx = \int_0^\pi (\sin^2 x)^2\,dx$$

$$= \int_0^\pi (\frac{1-\cos(2x)}{2})^2\,dx$$

$$= \frac{1}{4}\int_0^\pi (1 - 2\cos(2x) + \cos^2(2x))\,dx$$

$$= \frac{1}{4}\int_0^\pi (1 - 2\cos(2x) + \frac{1}{2} + \frac{1}{2}\cos(4x))\,dx$$

$$= \frac{1}{4}\int_0^\pi (\frac{3}{2} - 2\cos(2x) + \frac{1}{2}\cos(4x)\,dx$$

$$= \frac{1}{4}(\frac{3x}{2} - \sin(2x) + \frac{1}{8}\sin(4x))|_0^\pi$$

$$= \frac{3\pi}{8}.$$

11. We use integration by parts: Let $u = e^t$ and $dv = \cos t\,dt$. Then, $du = e^t\,dt$ and $v = \sin t$. Thus,

$$\int e^t \cos t\,dt = e^t \sin t - \int e^t \sin t\,dt.$$

We use integration by parts again. Let $u = e^t$ and $dv = \sin t\,dt$. Then, $du = e^t\,dt$ and $v = -\cos t$. The above becomes

$$\int e^t \cos t\,dt = e^t \sin t + e^t \cos t - \int e^t \cos t\,dt.$$

By moving the integral on the right to the left hand side, we have

$$2\int e^t \cos t\,dt = e^t \sin t + e^t \cos t + C$$

or

$$\int e^t \cos t\,dt = \frac{1}{2}e^t \sin t + \frac{1}{2}e^t \cos t + C.$$

12. There are a number of ways to do this problem correctly, depending on where you set the origin. In the derivation below, I will assume $y = 0$ at the bottom of the cone, and that $y = 10$ at the top. At a height y, I need to know the cross sectional area of the cone. Since the cross sectional radius is 0 at the bottom and 4 at the top, the cross sectional radius at

height y is $r = \frac{2}{5}y$. (You can find this by similar triangles or by finding the linear function from $(0,0)$ to $(10,4)$.) The water at height y needs to travel a distance of $10 - y$ to get to the top of the tank. Since there is only 8 feet of water in the tank, we integrate from 0 to 8. The integral is therefore

$$60\int_0^8 \pi(\frac{2}{5}y)^2(10-y)\,dy = \frac{48}{5}\int_0^8 (10y^2 - y^3)\,dy.$$

13. We can separate this question into a question about the bucket and a question about the water.

    First, since the bucket weighs 4 lb and is lifted 80 feet, the work is $4 \times 80 = 320$ ft-lbs.

    Second, since the water is leaking out, we have a variable force. At $y = 0$ (bottom of the well), the force is $F = 40$. At $y = 80$, the force is $F = 32$. Since the water is leaking out at a constant rate, we can assume the force function is linear. The slope is $(32 - 40)/(80 - 0) = -1/10$. Thus, the force function is $F(y) = -\frac{1}{10}y + 40$. The work done in lifting the water is

    $$\int_0^8 0 - \frac{1}{10}y + 40\,dy = -\frac{1}{20}y^2 + 40y|_0^{80}$$

    $$= -\frac{1}{20}6400 + 40 \cdot 80 = 2880\ ft - lbs$$

    Thus the total work is 3200 ft-lbs.